Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 3, pp. 369–377, 1995

0021–8928/95 \$24.00+0.00

0021–8928(95)00044-5

ASYMPTOTIC OPTIMIZATION OF LINEAR DYNAMICAL SYSTEMS WITH CONTROLS OF DIFFERENT STRENGTHS†

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(*Received* 14 *March* 1994)

The terminal problem for the optimal control (OC) of a linear system with constant coefficients and two controls whose level of influence on the system are very different is considered. In the first section the basic problem-optimization methods, previously developed by the authors, are used to construct solutions for systems with a small controlling perturbation (weak control). In the second section a system with a large coefficient in front of the control (strong control) is considered and its solution is also constructed using the solution of the basic problem.

The algorithms presented below for the asymptotic solution of optimization problems for dynamical systems with controlling forces of differing strengths follow the classical perturbation scheme. In essence they reduce the original problem to a simpler, basic problem and a relatively straightforward correction to the solution of the latter. Other kinds of optimal control problems that can be efficiently solved by the method used below are given in [1–6].

1. A SYSTEM WITH WEAK CONTROL

In the class of piecewise-continuous controlling actions $u(t)$, $\upsilon(t)$, $t \in T = [0, t_*]$ we consider the following problem for the optimal control (OC) of a linear stationary system

$$J(u, \upsilon) = \mathbf{c}'\mathbf{x}(t_*) \to \max \tag{1.1}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u + \mu \mathbf{b}_2 \upsilon, \quad \mathbf{x}(0) = \mathbf{x}^0 \tag{1.2}$$

$$|u(t)| \leqslant 1, \quad |\upsilon(t)| \leqslant 1, \quad t \in T \tag{1.3}$$

$$\mathbf{H}\mathbf{x}(t_*) = \mathbf{g} \tag{1.4}$$

where μ is a small positive parameter, u, υ are scalars, $\mathbf{x}$ is an n-vector, $\mathbf{g}$ is an m-vector ($m < n$), and the remaining elements of the problem have the appropriate dimensions. We assume that rank $\mathbf{H} = m$.

The piecewise-continuous functions $u(t, \mu)$, $\upsilon(t, \mu)$, $t \in T$ are called admissible controls in the problem under consideration if they and the trajectories of system (1.2) that they generate satisfy conditions (1.3) and (1.4). The admissible control for which the quality criterion $J(u, \upsilon)$ reaches its maximum value is called the OC. Along with these familiar concepts we shall define what we mean by asymptotic approximations to the solution of this problem.

Definitions. The family of piecewise-continuous functions $u(t, \mu)$, $\upsilon(t, \mu)$, $t \in T$ is said to be an asymptotically k-admissible control if they satisfy condition (1.3) and if the trajectories $x(t, \mu)$, $t \in T$ of system (1.2) which they generate satisfy the termination condition (1.4) with accuracy to $O(\mu^{k+1})$. An admissible (asymptotically k-admissible) control is said to be asymptotically s-optimal if the discrepancy of its quality criterion from that of the optimal control is of order $O(\mu^{s+1})$.

This section presents an algorithm which for any given natural number s enables one to construct an asymptotically s-admissible s-optimal control for the problem under consideration. It also describes a numerical procedure which uses this asymptotic approximation to solve problem (1.1)–(1.4) exactly for a given value of the small parameter.

†*Prikl. Mat. Mekh.* Vol. 59, No. 3, pp. 391–400, 1995.

The first stage of the algorithm consists of solving the following terminal control problem

$$J_0(u) = \mathbf{c}'\mathbf{x}(t_*) \to \max, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u, \quad \mathbf{x}(0) = \mathbf{x}^0$$
$$|u(t)| \leqslant 1, \quad t \in T, \quad \mathbf{H}\mathbf{x}(t_*) = \mathbf{g} \tag{1.5}$$

which we shall call the basic problem.

Assumption 1.1 Problem (1.5) has a solution and is "simple" [7].

Then, solving it using the direct support method [1], we obtain the following:

1. the OC and trajectory $u^0(t)$, $\mathbf{x}^0(t)$, $t \in T$;
2. the optimal support $\{\eta_1, \ldots, \eta_m\}$, i.e. a set of m points in the interval $]0, t_*[$ such that the $(m \times m)$ matrix

$$\mathbf{\Phi}_1 = (\varphi_1(\eta_j), \quad j = 1, 2, \ldots, m) \tag{1.6}$$

which is known as the support matrix, is non-degenerate, where

$$\varphi_1(t) = \mathbf{H}\mathbf{F}(t)\mathbf{b}_1, \quad t \in T \tag{1.7}$$

and $\mathbf{F}(t)$, $t \in T$ is an $(n \times n)$ matrix function satisfying the differential equation

$$\dot{\mathbf{F}} = -\mathbf{F}\mathbf{A}, \quad \mathbf{F}(t_*) = \mathbf{E}; \tag{1.8}$$

3. the vector of potentials $\lambda_0' = \mathbf{c}'_1\mathbf{\Phi}_1^{-1}$, where $\mathbf{c}_1 = (\gamma_1(\eta_j), j = 1, 2, \ldots, m)'$, $\gamma_1(t) = \mathbf{c}'F(t)\mathbf{b}_1$, $t \in T$;
4. the cocontrol $\Delta_1(t) = \psi_0'(t)\mathbf{b}_1$, $t \in T$ constructed from the solution $\psi_0(t)$, $t \in T$ of the conjugate system $\dot{\psi}_0 = -\mathbf{A}'\psi_0$, $\psi_0(t_*) = \mathbf{c} - \mathbf{H}'\lambda_0$.

The cocontrol is related to the optimal control by the relation $u^0(t) = \operatorname{sgn} \Delta_1(t)$, $t \in T$ and possesses the following property: $\Delta_1(\eta_j) = 0$, $\dot{\Delta}_1(\eta_j) \neq 0$ $(j = 1, 2, \ldots, m)$. We shall denote all the zeros of the cocontrol by $t_{01}, \ldots, t_{0l}$, arranging them in increasing order. Because they include the support times, we have $l \geqslant m$.

Assumption 1.2. $t_{0j} \in\ ]0, t_*[$, $\dot{\Delta}_1(t_{0j}) \neq 0$ $(j = 1, 2, \ldots, l)$.

Then the times $t_{01}, \ldots, t_{0l}$ are precisely the switching times for the OC.

Remark. The choice of the direct support method to solve the basic problem is explained by the fact that as well as giving the OC, it also gives additional information for constructing the asymptotic limit. Moreover, the algorithm to be described and the direct support method can be implemented with almost the same assumptions about the basic problem. The direct support method is applicable to "simple" problems. Hence, if problem (1.5) can be solved by these methods, then Assumption 1.1 is satisfied. The basic problem does not of course have to be solved by the direct support method. If another method is used, it is convenient to replace Assumption 1.1 with the following: the OC in problem (1.5) has at least m switching points, which include the points $\eta_1, \ldots, \eta_m$ for which matrix (1.6) is non-degenerate.

After solving the basic problem we find the zeros $\tau_{01}, \ldots, \tau_{0p}$ of the function $\Delta_2(t) = \psi_0(t)\mathbf{b}_2$, $t \in T$, numbered in increasing order.

Assumption 1.3. If $p \geqslant 1$ then $\tau_{0i} \in\ ]0, t_*[$, $\dot{\Delta}_2(\tau_{0i}) \neq 0$, $i = 1, 2, \ldots, p$.

We introduce the numbers $\alpha_0, \alpha_1, \ldots, \alpha_l, \beta_0, \beta_1, \ldots \beta_p$: $\alpha_0 = \operatorname{sgn} \Delta_1(0)$, $\alpha_j = -\alpha_{j-1}$ $(j = 1, 2, \ldots, l)$; $\beta_0 = \operatorname{sgn} \Delta_2(0)$, $\beta_i = -\beta_{i-1}$ $(i = 1, 2, \ldots, p)$. We denote by $\psi(t, \lambda)$, $t \in T$, $\lambda \in R^m$ the trajectory of the conjugate system

$$\dot{\psi} = -\mathbf{A}'\psi, \quad \psi(t_*) = \mathbf{c} - \mathbf{H}'\lambda \tag{1.9}$$

Suppose further that $t_1, \ldots, t_l, \tau_1, \ldots, \tau_p$ are numbers in the interval $]0, t_*[$ such that $t_1 < t_2 < \ldots < t_l$, $\tau_1 < \tau_2 < \ldots < \tau_p$. We denote by $u(t, t_1, \ldots, t_l)$, $\upsilon(t, \tau_1, \ldots, \tau)$, $t \in T$ a two-level controlling action that switches at the points $t_1, \ldots, t_l, \tau_1, \ldots, \tau_p$ and takes the values α_0, β_0, respectively, in the first constancy interval. The trajectory of system (1.2) generated by this control is denoted by

$\mathbf{x}(t, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu), t \in T$. Subsequent calculations are based on the assertion formulated below.

Theorem 1. If Assumptions 1.1–1.3 are satisfied, then for sufficiently small μ the OC in problem (1.1)–(1.4) has the form

$$u^0(t, \mu) = u(t, t_1(\mu), \dots, t_l(\mu)), \quad \upsilon^0(t, \mu) = \upsilon(t, \tau_1(\mu), \dots, \tau_p(\mu)) \tag{1.10}$$

The switching points of the OC and the associated normal vector of Lagrange multipliers $\lambda(\mu)$ solve the system of equations

$$\mathbf{Hx}(t_*, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) - \mathbf{g} = \mathbf{0} \tag{1.11}$$

$$\psi'(t_j, \lambda)\mathbf{b}_1 = 0, \quad j = 1, 2, \dots, l, \quad \psi'(\tau_j, \lambda)\mathbf{b}_2 = 0, \quad (i = 1, 2, \dots, p)$$

and can be expanded asymptotically

$$t_j(\mu) \sim \sum_{k=0}^{\infty} \mu^k t_{kj}, \quad j = 1, 2, \dots, l; \quad \tau_j(\mu) \sim \sum_{k=0}^{\infty} \mu^k \tau_{ki}$$

$$i = 1, 2, \dots, p; \qquad \lambda(\mu) \sim \sum_{k=0}^{\infty} \mu^k \lambda_k \tag{1.12}$$

Proof. Let $\mathbf{N}(t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) = \mathbf{Hx}(t_*, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) - \mathbf{g}$. By the Cauchy formula we have

$$\begin{aligned}\mathbf{N}(t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) &= \mathbf{N}_0(t_1, \dots, t_l) + \mu\mathbf{N}_1(\tau_1, \dots, \tau_p)\\ &= \mathbf{HF}(0)\mathbf{x}^0 + \alpha_0 \int_0^{t_1} \varphi_1(t)dt + \dots + \alpha_l \int_{t_l}^{t_*} \varphi_1(t)dt - \mathbf{g}\\ &+\mu\left(\beta_0 \int_0^{\tau_1} \varphi_2(t)dt + \dots + \beta_p \int_{\tau_p}^{t_*} \varphi_2(t)dt\right)\end{aligned} \tag{1.13}$$

where $\mathbf{F}(t)$, $t \in T$ are matrix functions which solve Eq. (1.8), $\varphi_1(t)$, $t \in T$ are given by formula (1.7) and

$$\varphi_2(t) = \mathbf{HF}(t)\mathbf{b}_2, \quad t \in T \tag{1.14}$$

To shorten the equations we introduce the vectors

$$\mathbf{h}' = (t_1, \dots, t_l, \tau_1, \dots, \tau_p, \lambda'), \quad \mathbf{h}'_0 = (t_{01}, \dots, t_{0l}, \tau_{01}, \dots, \tau_{0p}, \lambda'_0) \tag{1.15}$$

System (1.11) can be written in the form

$$\mathbf{R}(\mathbf{h}, \mu) = \mathbf{0}$$

$$\mathbf{R}(\mathbf{h}, \mu) = \left\| \begin{matrix} \mathbf{N}(t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) & \\ \psi'(t_j, \lambda)\mathbf{b}_1 & j = 1, 2, \dots, l \\ \psi'(t_i, \lambda)\mathbf{b}_2 & i = 1, 2, \dots, p \end{matrix} \right\| \tag{1.16}$$

According to formula (1.13) we have

$$\mathbf{R}(\mathbf{h}, \mu) = \mathbf{R}_0(\mathbf{h}) + \mu\mathbf{R}_1(\mathbf{h})$$

$$\mathbf{R}_0(\mathbf{h}) = \left\| \begin{matrix} \mathbf{N}_0(t_1, \dots, t_l) & \\ \psi'(t_j, \lambda)\mathbf{b}_1, & j = 1, 2, \dots, l \\ \psi'(\tau_i, \lambda)\mathbf{b}_2, & i = 1, 2, \dots, p \end{matrix} \right\|, \quad \mathbf{R}_1(\mathbf{h}) = \left\| \begin{matrix} \mathbf{N}_1(\tau_1, \dots, \tau_p) \\ 0 \end{matrix} \right\|$$

We put $\mathbf{R}(\mathbf{h}, 0) = \mathbf{R}_0(\mathbf{h})$. Then the vector function $\mathbf{R}(\mathbf{h}, \mu)$ is continuous together with its partial derivatives with respect to the components of the vector $\mathbf{h}$ in the domain $\| \mathbf{h} - \mathbf{h}_0 \| < \varepsilon$, $0 \leqslant \mu < \mu_0$, where ε, μ_0 are sufficiently small positive numbers.

The control $u^0(t)$, $t \in T$ is admissible in the basic problem, and so $\mathbf{N}_0(t_{01}, \ldots, t_{0l}) = \mathbf{H}\mathbf{x}^0(t_*) - \mathbf{g} = 0$. Because $\psi(t, \lambda_0) = \psi_0(t)$, $t \in T$ we have $\psi'(t_{0j}, \lambda_0)\mathbf{b}_1 = \Delta_1(t_{0j}) = 0$ $(j = 1, 2, \ldots, l)$; $\psi'(\tau_{0i}, \lambda_0)\mathbf{b}_2 = \Delta_2(\tau_{0i}) = 0$ $(i = 1, 2, \ldots, p)$. Thus $\mathbf{R}(\mathbf{h}_0, 0) = \mathbf{R}_0(\mathbf{h}_0) = 0$.

We verify by direct differentiation that the Jacobi matrix has the form

$$\mathbf{I}_1 = \frac{\partial \mathbf{R}(\mathbf{h}_0, 0)}{\partial \mathbf{h}} = \frac{\partial \mathbf{R}_0(\mathbf{h}_0)}{\partial \mathbf{h}} = \begin{Vmatrix} \mathbf{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{B}_3 \\ \mathbf{0} & \mathbf{B}_4 & \mathbf{B}_5 \end{Vmatrix} \tag{1.17}$$

where

$$\begin{gathered} \mathbf{B}_1 = -(2\alpha_j\varphi_1(t_{0j}), \quad j = 1, 2, \ldots, l, \quad \mathbf{B}_2 = \operatorname{diag}(\dot{\Delta}_1(t_{0j}), \quad j = 1, 2, \ldots, l) \\ \mathbf{B}_3 = -(\varphi_1(t_{0j}), \quad j = 1, 2, \ldots, l)' \\ \mathbf{B}_4 = \operatorname{diag}(\dot{\Delta}_2(\tau_{0i}), \quad i = 1, 2, \ldots, p), \quad \mathbf{B}_5 = -(\varphi_2(\tau_{0i}), \quad i = 1, 2, \ldots, p)' \end{gathered} \tag{1.18}$$

The matrix $(\varphi_1(t_{0j}), j = 1, 2, \ldots, l)$ has complete rank because it contains the non-degenerate support matrix (1.6) as a submatrix. It follows from this and from Assumptions 1.2 and 1.3 that the Jacobi matrix (1.17) is non-degenerate.

Hence system (1.16), or equivalently, (1.11), satisfies all the conditions of the implicit function theorem. According to this theorem, in some right-sided neighbourhood of zero $0 \leqslant \mu < \mu_1$ there are uniquely defined continuous functions $t_j(\mu)$ $(j = 1, 2, \ldots, l)$, $\tau_j(\mu)$ $(i = 1, 2, \ldots, p)$ $\lambda(\mu)$ satisfying system (1.11) such that $t_j(0) = t_{0j}$ $(j = 1, 2, \ldots, l)$, $\tau_i(0) = \tau_{0i}$ $(i = 1, 2, \ldots, p)$, $\lambda(0) = \lambda_0$.

In other words, for sufficiently small μ, problem (1.1)–(1.4) has an admissible control $u^0(t, \mu)$, $\upsilon^0(t, \mu)$, $t \in T$ of the form (1.10) and a Lagrange vector $\lambda(\mu)$ such that the switching points $u^0(t, \mu)$, $t \in T$ and $\upsilon^0(t, \mu)$, $t \in T$ are, respectively, zeros of the functions $\Delta_1(t, \mu) = \psi'(t, \mu)\mathbf{b}_1$, $\Delta_2(t, \mu) = \psi'(t, \mu)\mathbf{b}_2$, $t \in T$ constructed from the solution $\psi(t, \mu)$, $t \in T$ of the conjugate system (1.9) with $\lambda = \lambda(\mu)$.

Because $\mathbf{R}_0(\mathbf{h})$, $\mathbf{R}_1(\mathbf{h})$ are infinitely-differentiable functions, we have the asymptotic expansions (1.12). Using Assumptions (1.2) and (1.3) together with the implicit function theorem we verify that the cocontrol $\Delta_1(t, \mu)$, $t \in T$ vanishes only at the point $t_j(\mu)$ $(j = 1, 2, \ldots, l)$, while $\Delta_2(t, \mu)$, $t \in T$ has no zeros other than $\tau_i(\mu)$ $(i = 1, 2, \ldots, p)$, with $u^0(t, \mu) = \operatorname{sgn} \Delta_1(t, \mu)$, $u^0(t, \mu) = \operatorname{sgn} \Delta_2(t, \mu)$, $t \in T$. The latter means that an admissible control $u^0(t, \mu)$, $\upsilon^0(t, \mu)$, $t \in T$ satisfies the Pontryagin maximum principle [8] with the normal Lagrange multiplier vector $\lambda(\mu)$, and is therefore the OC. The theorem is proved.

We choose a positive integer s. In order to construct an asymptotically s-admissible s-optimal control for problem (1.1)–(1.4) it is sufficient to find the polynomials

$$t_j^{(s)}(\mu) = \sum_{k=0}^{s} \mu^k t_{kj}, \quad j = 1, 2, \ldots, l; \quad \tau_j^{(s-1)}(\mu) = \sum_{k=0}^{(s-1)} \mu^k \tau_{ki}, \quad i = 1, 2, \ldots, p \tag{1.19}$$

This can be done as follows. Let

$$\mathbf{h}_k' = (t_{k1}, \ldots, t_{kl}, \ \tau_{k1}, \ldots, \tau_{kp}, \lambda_k'), \quad \mathbf{h}_s(\mu) = \sum_{k=0}^{s} \mu^k \mathbf{h}_k$$

We expand the vector function $\mathbf{R}(\mathbf{h}_s(\mu), \mu)$ in powers of μ to order s inclusive using Taylor's formula and equate the coefficients of the expansions to zero (beginning with the coefficient of μ). As a result we obtain non-degenerate systems of linear equations which sequentially determine the vectors $\mathbf{h}_k$ $(k = 1, 2, \ldots, s)$

$$\mathbf{I}_1\mathbf{h}_1 = -\mathbf{R}_1(\mathbf{h}_0), \quad \mathbf{I}_1\mathbf{h}_2 = -\frac{\partial \mathbf{R}_1}{\partial \mathbf{h}}(\mathbf{h}_0)\mathbf{h}_1 - \frac{1}{2}\mathbf{h}_1' \frac{\partial^2 \mathbf{R}_0}{\partial \mathbf{h}^2}(\mathbf{h}_0)\mathbf{h}_1, \ldots \tag{1.20}$$

We note that in view of the structure (1.17) of the Jacobi matrix $\mathbf{I}_1$ these systems decouple. In particular, the vector λ_1 is found as a solution of the system $\mathbf{B}_1\mathbf{B}_2^{-1}\mathbf{B}_3\lambda_1 = \mathbf{N}_1(\tau_{01}, \ldots, \tau_{0p})$, and $t_{1j} = \lambda_1\varphi_1(t_{0j})/\dot{\Delta}_1(t_{0j})$ $(j = 1, 2, \ldots, l)$.

Solving system (1.20) sequentially we find the vectors $\mathbf{h}_k$ $(k = 1, 2, \ldots, s)$ and construct the polynomials (1.19). The control $u_s(t, \mu) = u(t, t_1^{(s)}(\mu), \ldots, t_l^{(s)}(\mu)), \upsilon_{s-1}(t, \mu) = u(t, \tau_1^{(s-1)}(\mu), \ldots, \tau_p^{(s-1)}(\mu)), t \in T$ will obviously be an asymptotically s-admissible, s-optimal control for problem (1.1)–(1.4).

The constructed asymptotic approximations to the roots of system (1.6) can be used to solve this system numerically, which means, in the problem under considerations, for a specified value of μ. This requires the use of a refining procedure [1], i.e. using Newton's method for finding the roots of system (1.16) taking $\mathbf{h}_s(\mu)$ as the initial approximation.

2. A SYSTEM WITH STRONG CONTROL

In the class of piecewise-continuous controls $u(t), \upsilon(t), t \in T = [0, t_*]$ we consider the optimal control problem

$$J(u, \upsilon) = \mathbf{c}'\mathbf{x}(t_*) \to \max, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u + \mathbf{b}_2\upsilon/\mu, \quad \mathbf{x}(0) = \mathbf{x}^0$$
$$|u(t)| \leqslant 1, \quad |\upsilon(t)| \leqslant 1, \quad t \in T \quad \mathbf{H}\mathbf{x}(t_*) = \mathbf{g} \tag{2.1}$$

where, as before, μ is a small positive parameter, u, υ are scalars, $\mathbf{x}$ is an n-vector, and $\mathbf{g}$ is an m-vector $(m < n)$. We assume that rank $\mathbf{H} = m$.

We will describe an algorithm which for any specified positive integer s enables us to construct an asymptotically s-admissible s-optimal control for problem (2.1). This control is defined in the same way as for problem (1.1)–(1.4).

The notation in this section is independent of that of the preceding one: the same symbol may in general denote different quantities. However, there is an analogy between quantities for which we use the same symbol.

In this case the basic problem has the form

$$J_0(\upsilon) = \mathbf{c}'\mathbf{x}(t_*) \to \max, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_2\upsilon, \quad \mathbf{x}(0) = 0$$
$$|\upsilon(t)| \leqslant 1, \quad t \in T, \quad \mathbf{H}\mathbf{x}(t_*) = 0 \tag{2.2}$$

Assumption 2.1. Problem (2.2) is "simple".

Solving this problem by the direct support method we obtain

1. the OC and the trajectory $\upsilon^0(t), \mathbf{x}^0(t), t \in T$;
2. the optimal support $\{\sigma_1, \ldots, \sigma_m\}$ and the associated non-degenerate support matrix

$$\mathbf{\Phi}_2 = (\varphi_2(\sigma_i), \quad i = 1, 2, \ldots, m) \tag{2.3}$$

where $\varphi_2(t), t \in T$ is the m-vector function defined by formula (1.14);

3. the vector of the potentials $\lambda_0' = \mathbf{c}_2'\mathbf{\Phi}_2^{-1}$, where $\mathbf{c}_2 = (\gamma_2(\sigma_i), i = 1, 2, \ldots, \mathbf{m})'$, $\gamma_2(t) = \mathbf{c}'\mathbf{F}(t)\mathbf{b}_2$, $t \in T$, and the $(n \times n)$ matrix function $\mathbf{F}(t), t \in T$ satisfies Eq. (1.8);

4. the cocontrol $\Delta_2(t) = \psi_0(t)\mathbf{b}_2, t \in T$ constructed from the solution $\psi_0(t), t \in T$ of the conjugate system $\dot{\psi}_0 = -\mathbf{A}'\psi_0, \psi_0(t_*) = \mathbf{c} - \mathbf{H}'\lambda_0$.

The cocontrol, coupled to the OC through the relation $\upsilon^0(t) = \operatorname{sgn}\Delta_2(t), t \in T$, has the property $\Delta_2(\sigma_i) = 0, \dot{\Delta}_2(\sigma_i) \neq 0, i = 1, 2, \ldots, m$.

Remark. Assumption 2.1 can be replaced by the following: the OC in problem (2.2) has at least m switching points, including the points $\sigma_1, \ldots, \sigma_m$ at which matrix (2.3) is non-degenerate.

Suppose $\tau_{01}, \ldots, \tau_{0p}$ are all the zeros of the cocontrol, in increasing order. It is clear that $p \geqslant m$.

Assumption 2.2. $\tau_{0i} \in]0, t_*[, \dot{\Delta}_1(\tau_{0j}) \neq 0$ $(i = 1, 2, \ldots, p)$.

Then $\tau_{01}, \ldots, \tau_{0p}$ are the switching points of the control $\upsilon^0(t), t \in T$.

After solving the basic problem we find the zeros $t_{01}, \ldots, t_{0l}$ of the function $\Delta_1(t) = \psi_0(t)\mathbf{b}_1$, in increasing order.

Assumption 2.3. If $l \geqslant 1$, then $t_{0j} \in\,]0, t_*[$, $\dot{\Delta}_1(t_{0j}) \neq 0$ $(j = 1, 2, \dots, l)$.

We introduce the numbers $\alpha_0, \alpha_1, \dots, \alpha_l$. $\beta_0, \beta_1, \dots, \beta_p$ and two-level controls $u(t, t_1, \dots, t_l)$, $\upsilon(t, \tau_1, \dots, \tau_p)$, $t \in T$ as in the preceding section. Suppose, as before, that $\psi(t, \lambda)$, $t \in T$, $\lambda \in \mathbf{R}^m$ are trajectories of the conjugate system (1.9). We denote by $\mathbf{x}(t, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu)$, $t \in T$ the trajectories of the dynamical system in problem (2.1) generated by the control $u(t, t_1, \dots, t_l)$, $\upsilon(t, \tau_1, \dots, \tau_p)$, $t \in T$. The algorithm for constructing the asymptotic behaviour of the solution of problem (2.1) is based on the following assertions.

Theorem 2. When Assumptions 2.1–2.3 are satisfied, the OC for problem (2.1) is of the form (1.10) for sufficiently small μ. The switching points for this OC and the associated normal vector of Lagrange multipliers $\lambda(\mu)$ satisfy the system of equations

$$\begin{gathered}\mu(\mathbf{Hx}(t_*, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) - \mathbf{g} = 0\\ \psi'(t_j, \lambda)\mathbf{b}_1 = 0, \quad j = 1, 2, \dots, l, \quad \psi'(\tau_i, \lambda)\mathbf{b}_2 = 0, \quad i = 1, 2, \dots, p\end{gathered} \tag{2.4}$$

and have an asymptotic expansion (1.12).

Proof. We put $\mathbf{K}(t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) = \mu(\mathbf{Hx}(t_*, t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) - \mathbf{g})$. Applying the Cauchy formula we obtain

$$\begin{aligned}&\mathbf{K}(t_1, \dots, t_l, \tau_1, \dots, \tau_p, \mu) = \mathbf{K}_0(\tau_1, \dots, \tau_p) + \mu\mathbf{K}_1(t_1, \dots, t_l)\\ &= \beta_0\int_0^{\tau_1}\varphi_2(t)dt + \dots + \beta_p\int_{\tau_p}^{t_*}\varphi_2(t)dt + \mu(\mathbf{HF}(0)\mathbf{x}^0\\ &+ \alpha_0\int_0^{t_1}\varphi_1(t)dt + \dots + \alpha_l\int_{t_l}^{t_*}\varphi_1(t)dt - \mathbf{g})\end{aligned} \tag{2.5}$$

where $\mathbf{F}(t)$, $t \in T$ is the solution of Eq. (1.8) and $\varphi_1(t)$, $\varphi_2(t)$, $t \in T$ are given by formulae (1.7) and (1.14).

We introduce the vectors (1.15) and write system (2.4) in the form

$$\mathbf{P}(\mathbf{h}, \mu) = \mathbf{0} \tag{2.6}$$

$$\mathbf{P}(\mathbf{h}, \mu) = \begin{Vmatrix}\mathbf{K}(t_1, \dots, t_l, \ \tau_1, \dots, \tau_p, \mu) & \\ \psi'(t_j, \lambda)\mathbf{b}_1, & j = 1, 2, \dots, l\\ \psi'(\tau_i, \lambda)\mathbf{b}_2, & i = 1, 2, \dots, p\end{Vmatrix}$$

From (2.5)

$$\mathbf{P}(\mathbf{h}, \ \mu) = \mathbf{P}_0(\mathbf{h}) + \mu\mathbf{P}_1(\mathbf{h})$$

$$\mathbf{P}_0(\mathbf{h}) = \begin{Vmatrix}\mathbf{K}_0(\tau_1, \ \dots, \ \tau_p) & \\ \psi'(t_j, \ \lambda)\mathbf{b}_1 & j = 1, \ 2, \ \dots, \ l\\ \psi'(\tau_i, \ \lambda)\mathbf{b}_2 & i = 1, \ 2, \ \dots, \ p\end{Vmatrix}, \quad \mathbf{P}_1(\mathbf{h}) = \begin{Vmatrix}\mathbf{K}_1(t_1, \ \dots, \ t_l)\\ 0\end{Vmatrix}$$

We define $\mathbf{P}(\mathbf{h}, 0) = \mathbf{P}_0(\mathbf{h})$. Then the vector function $\mathbf{P}(\mathbf{h}, \mu)$ is continuous together with all its partial derivative with respect to the components of the vector $\mathbf{h}$ in the domain $\| \mathbf{h} - \mathbf{h}_0 \| < \varepsilon$, $0 \leqslant \mu < \mu_0$, where ε, μ_0 are some sufficiently small positive numbers.

Because the control $\upsilon^0(t)$, $t \in T$ is admissible in problem (2.2) and $\psi(t, \lambda_0) = \psi_0(t)$, $t \in T$, we have $\mathbf{P}(\mathbf{h}_0, 0) = \mathbf{P}_0(\mathbf{h}_0) = 0$. The Jacobi matrix of system (2.6) has the form

$$\mathbf{I}_2 = \frac{\partial\mathbf{P}}{\partial\mathbf{h}}(\mathbf{h}_0, \ 0) = \begin{Vmatrix}\mathbf{0} & \mathbf{B} & \mathbf{0}\\ \mathbf{B}_2 & \mathbf{0} & \mathbf{B}_3\\ \mathbf{0} & \mathbf{B}_4 & \mathbf{B}_5\end{Vmatrix} \tag{2.7}$$

where $\mathbf{B} = -(2\beta_i\varphi_2(\tau_{0i})$, $i = 1, 2, \dots, p)$. The remaining blocks of the matrix are given by (1.18).

The matrix $(\varphi_2(\tau_{0i}), i = 1, 2, \ldots, p)$ has complete rank because it has the non-degenerate matrix (2.3) as a submatrix. From this and from Assumptions 2.2 and 2.3 it follows that the matrix $\mathbf{I}_2$ is non-degenerate.

All the conditions of the implicit function theorem are therefore satisfied by system (2.6). The proof of Theorem 2 is completed in the same way as for Theorem 1.

We now consider the algorithm for constructing an s-admissible s-optimal control for Problem 2.1. Using the method of undetermined coefficients described in the preceding section we construct non-degenerate systems of linear equations

$$\mathbf{I}_2\mathbf{h}_1 = -\mathbf{P}_1(\mathbf{h}_0), \quad \mathbf{I}_2\mathbf{h}_2 = -\frac{\partial \mathbf{P}_1}{\partial \mathbf{h}}(\mathbf{h}_0)\mathbf{h}_1 - \frac{1}{2}\mathbf{h}_1'\frac{\partial^2 \mathbf{P}_0}{\partial \mathbf{h}^2}(\mathbf{h}_0)\mathbf{h}_1, \ldots \tag{2.8}$$

for the sequential calculation of the vectors $\mathbf{h}'_k = (t_{k1}, \ldots, t_{kl}, \tau_{k1}, \ldots, \tau_{kp}, \lambda'_k)$ $(k = 1, 2, \ldots, s + 1)$. Because $\mathbf{I}_2$ has the matrix structure (2.7) this system decouples. Thus the vector λ_1 satisfies the system $\mathbf{B}\mathbf{B}_4^{-1}\mathbf{B}_5\lambda_1 = \mathbf{K}_1(t_{01}, \ldots, t_{0l})$, and $\tau_{1i} = \lambda'_1\, \varphi_2(\tau_{0i})/\dot{\Delta}_2(\tau_{0i})$ $(i = 1, 2, \ldots, p)$.

Solving system (2.8) sequentially we find the vectors $\mathbf{h}_k$ $(k = 1, 2, \ldots, s + 1)$ and construct the polynomials

$$t_j^{(s)}(\mu) = \sum_{k=0}^{s} \mu^k t_{kj}, \quad j = 1, 2, \ldots, l; \quad \tau_i^{(s-1)}(\mu) = \sum_{k=0}^{(s-1)} \mu^k \tau_{ki}, \quad i = 1, 2, \ldots, p$$

The control $u_s(t, \mu) = u(t, t^{(s)}{}_1(\mu), \ldots, t^{(s)}{}_l(\mu))$, $\upsilon_{s+1}(t, \mu) = \upsilon(t, \tau_1{}^{(s+1)}(\mu), \ldots, \tau_p{}^{(s+1)}(\mu))$, $t \in T$ is an asymptotically s-admissible s-optimal control for problem (2.1).

The constructed asymptotic approximations for the roots of system (2.4) can be used to solve problem (2.1) exactly for a specified value of μ by taking them to be the initial approximations in a refinement procedure [1].

3. EXAMPLE

Consider a problem of the form (1.1)–(1.4)

$$x_2(3) \to \max, \quad \dot{x}_1 = x_2 + u, \quad \dot{x}_2 = -x_1 + \mu\upsilon, \quad x_1(0) = x_2(0) = 1$$
$$|u(t)| \le 1, \quad |\upsilon(t)| \le 1, \quad t \in [0, 3], x_1(3) = 0 \tag{3.1}$$

which models the control process for the rotation of a dynamically symmetric rigid body by means of two torques. The OC for the basic problem switches at a single point $t_{01} = 2.482170$, taking the value -1 in the first constancy interval. (All calculations were performed to six decimal places.) The optimal control has the associated Lagrange multiplier $\lambda_0 = -0.569685$ and cocontrol $\Delta_1(t) = \sin(t - t_{01})/\cos(3 - t_{01})$, $t \in [0, 3]$. The function $\Delta_2(t) = \cos(t - t_{01})/\cos(3 - t_{01})$, $t \in [0, 3]$ vanishes at the unique point $\tau_{01} = 0.911373$. This case satisfies Assumptions 1.1–1.3. The asymptotically 1-admissible 1-optimal control for problem (3.1) constructed using the algorithm given in Section 1 has the form

$$u_1(t, \mu) = \begin{cases} -1, & t \in [0, t_1^{(1)}(\mu)[\\ 1, & t \in [t_1^{(1)}(\mu), 3] \end{cases} \qquad \upsilon_0(t, \mu) = \begin{cases} -1, & t \in [0, \tau_{01}[\\ 1, & t \in [\tau_{01}, 3] \end{cases}$$

where $t_1^{(1)}(\mu) = t_{01} + \mu t_{11}$ and $t_{11} = 0.575443$.

The refinement procedure was used to find the optimal control $u^0(t, \mu)$, $\upsilon^0(t, \mu)$, $t \in [0, 3]$ for problem (3.1) for the two values 0.1 and 0.01 of the small parameter. The controls $u^0(t, 0.1)$, $\upsilon^0(t, 0.1)$, $u^0(t, 0.01)$, $\upsilon^0(t, 0.01)$, $t \in [0, 3]$ switch at the points 2.533748, 0.962951, 2.487858 and 0.917061, respectively, with the value -1 in the first constancy interval. Note that $t_1^{(1)}(0.1) = 2.539714$, $t_1^{(1)}(0.01) = 2.487924$.

Consider the problem

$$x_2(3) \to \max, \quad \dot{x}_1 = x_2 + u, \quad \dot{x}_2 = -x_1 + \upsilon/\mu, \quad x_1(0) = x_2(0) = 1 \tag{3.2}$$
$$|u(t)| \le 1, \quad |\upsilon(t)| \le 1, \quad t \in [0, 3], \quad x_1(3) = 0$$

which differs from problem (3.1) only in that the control υ is strong rather than weak.

The optimal control of the basic problem for this case switches at the point $\tau_{01} = 1.434207$ and takes the value -1 in the first constancy interval. It corresponds to the Lagrange multiplier $\lambda_0 = 0.005004$ and cocontrol $\Delta_2(t) = \sin(t - \tau_{01})/\sin(3 - \tau_{01})$, $t \in [0, 3]$. The function $\Delta_1(t) = -\cos(t - \tau_{01})/\sin(3 - \tau_{01})$, $t \in [0, 3]$ only takes negative values. Assumptions 2.1–2.3 are satisfied. The asymptotically 0-admissible, 0-optimal control for problem (3.2), constructed with the help of the algorithm described in Section 2, has the form

$$u_1(t, \mu) = -1, \qquad t \in [0, 3], \qquad \upsilon_1(t, \mu) = \begin{cases} -1, & t \in [0, \tau_1^{(1)}(\mu)[\\ 1, & t \in [\tau_1^{(1)}(\mu), 3] \end{cases}$$

where $\tau_1^{(1)}(\mu) = \tau_{01} + \mu\tau_{11}$, $\tau_{11} = -0.495009$.

According to Theorem 2, for sufficiently small μ the OC problem (3.2) has the following structure

$$u^0(t, \mu) = -1, \qquad t \in [0, 3], \qquad \upsilon^0(t, \mu) = \begin{cases} -1, & t \in [0, \tau_1(\mu)[\\ 1, & t \in [\tau_1(\mu), 3] \end{cases} \tag{3.3}$$

where $\tau_1(\mu) = \tau_1^{(1)}(\mu) + O(\mu^2)$. In this case system (2.4) has the form

$$\begin{gathered} 1 + \cos 3 - 2\cos(\tau_1 - 3) + \mu \cos 3 = 0 \\ \cos(\tau_1 - 3) + \lambda \sin(\tau_1 - 3) = 0 \end{gathered} \tag{3.4}$$

The solution of this problem when $\mu = 0.01$ was found using the refinement procedure. It turned out that $\tau_1(0.01) = 1.429257$, $\lambda(0.01) = 0.000054$. For comparison $\tau_1^{(1)}(0.01) = 1.429257$, i.e. the switching point of the asymptotically 0-admissible 0-optimal control coincided to six decimal places with the OC switching point. When $\mu = 0.1$ system (3.4) has the following solution: $\tau_1^* = 1.384693$, $\lambda^* = -0.044540$. It follows from the maximum principle that for a control of the form (3.3) to be optimal it is necessary for the Lagrange multiplier $\lambda(\mu)$ to be non-negative. The sign of λ^* shows that the value $\mu = 0, 1$ is insufficiently small, and the OC for problem (3.2) has a structure different from that of (3.3) for this value of μ. The optimal control $u^0(t, 0.1)$ should have one switch point $t_1(0.1)$ close to the final moment. In this case the refinement equation has the form

$$1 + \cos 3 - 2\cos(\tau_1 - 3) + \mu(\cos 3 - 2\sin(t_1 - 3)) = 0$$

$$\sin(t_1 - 3) - \lambda \cos(t_1 - 3) = 0, \quad \cos(\tau_1 - 3) + \lambda \sin(\tau_1 - 3) = 0$$

where $\mu = 0.1$. Solving this system by Newton's method, using the initial approximations $t_1 = 3$, $\tau_1 = \tau_{01}$, $\lambda = \lambda_0$ for the roots, we obtain $t_1(0.1) = 2.959538$, $\tau_1(0.1) = 1.388742$, $\lambda(0.1) = -0.040484$. The control

$$u^0(t, 0.1) = \begin{cases} -1, & t \in [0, t_1(0.1)[\\ 1, & t \in [t_1(0.1), 3] \end{cases} \qquad \upsilon^0(t, 0.1) = \begin{cases} -1, & t \in [0, \tau_1(0.1)[\\ 1, & t \in [\tau_1(0.1), 3] \end{cases}$$

satisfies the maximum principle and is therefore the OC for problem (3.2) with $m = 0.1$. Note, for comparison, that $t_1(1)(0.1) = 1.384706$.

This research was carried out with the financial support of the Republic of Belarus Foundation for Basic Research (F61-264).

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Translated by R.L.Z.